MAXIMAL LINEABILITY OF THE SET OF CONTINUOUS SURJECTIONS

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Abstract. Let m, n be positive integers. In this short note we prove that the set of all continuous and surjective functions from $\mathbb{R}^m$ to $\mathbb{R}^n$ contains (excluding the 0 function) a $\mathfrak{c}$-dimensional vector space. This result is optimal in terms of dimension.

1. Preliminaries

Lately the study of the linear structure of certain subsets of surjective functions in $\mathbb{R}^{\mathbb{R}}$ (such as everywhere surjective functions, perfectly everywhere surjective functions, or Jones functions) has attracted the attention of several authors working on Real Analysis and Set Theory (see, e.g. [1, 2, 4, 6, 7]). The previously mentioned functions are, indeed, very "pathological": for instance an everywhere surjective function f in $\mathbb{R}^{\mathbb{R}}$ verifies that $f(I) = \mathbb{R}$ for every interval $I \subset \mathbb{R}$ and the other classes (perfectly everywhere surjective functions and Jones functions) are particular cases of everywhere surjective functions and, thus, with even "worse" behavior. It has been shown [5] that there exists a $2^{\mathfrak{c}}$-dimensional vector space every non-zero element of which is a Jones function and, thus, everywhere surjective (here, $\mathfrak{c}$ stands for the cardinality of $\mathbb{R}$). Of course, this previous result is optimal in terms of dimension since $\dim(\mathbb{R}^{\mathbb{R}}) = 2^{\mathfrak{c}}$. However, all the previous classes are nowhere continuous, thus, it is natural to ask about the set of continuous surjections. The aim of this short note is to prove, in a more general framework that of $\mathbb{R}^{\mathbb{R}}$, that (for every $m, n \in \mathbb{N}$) the set of continuous surjections from $\mathbb{R}^m$ onto $\mathbb{R}^n$ is $\mathfrak{c}$-lineable [1] (that is, it contains a $\mathfrak{c}$-dimensional vector space every non-zero element of which is a continuous surjective function from $\mathbb{R}^m$ onto $\mathbb{R}^n$). Since $\dim \mathcal{C}(\mathbb{R}^m, \mathbb{R}^n) = \mathfrak{c}$ we have that this result would be the best possible in terms of dimension, that is, the set of continuous surjections from $\mathbb{R}^m$ onto $\mathbb{R}^n$ is maximal lineable [3].

While there are many trivial examples of surjective continuous functions in $\mathbb{R}^{\mathbb{R}}$, coming up with a concrete example of a continuous surjective function from $\mathbb{R}$ onto $\mathbb{R}^2$ is a totally different story. The existence of a continuous surjection from $\mathbb{R}$ onto $\mathbb{R}^2$ (a *Peano type* function) can be found in [8, p. 42] or [9, p. 274]. Both references use the existence of a continuous surjection from $[0, 1]$ onto $[0, 1]^2$ (a *Peano curve* in $[0, 1]^2$ or a *space filling curve*). The existence of this curve is proved, for instance, in [8] invoking a result due to A. D. Alexandrov: there is a continuous surjection from the Cantor space $\mathcal{K}$ onto any arbitrary nonempty compact metric space (see [8, p. 40]); in [9, section 44] the construction of the Peano curve is done geometrically, and is a consequence of the completeness of the space $\mathcal{C}(X, M)$ of all continuous functions from a topological space X to a complete metric space M, considering $\mathcal{C}(X, M)$ with the uniform metric.

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2. The lineability of the set of continuous surjections from $\mathbb{R}^m$ to $\mathbb{R}^n$

Let m and n be positive integers. Throughout this note we shall denote

$$\mathcal{S}_{m,n} = \{f : \mathbb{R}^m \longrightarrow \mathbb{R}^n \; ; \; f \text{ is continuous and surjective}\}.$$

The following result shows that $\mathcal{S}_{m,n} \neq \varnothing$, and uses the fact that $\mathcal{S}_{1,2} \neq \varnothing$ ([8, p. 42]).

Proposition 2.1. *Let $m, n \in \mathbb{N}$. There exists a continuous surjection $f : \mathbb{R}^m \to \mathbb{R}^n$.*

Proof. Let us take $f \in \mathcal{S}_{1,2}$. If $f_i := \pi_i \circ f$, $i = 1, 2$ denotes the i-coordinates functions of f ($f = (f_1, f_2)$), then the map $id_{\mathbb{R}} \times f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ defined by $id_{\mathbb{R}} \times f(t,s) := (t, f_1(s), f_2(s))$ is a continuous surjection. Thus, $(id_{\mathbb{R}} \times f) \circ f$ is in $\mathcal{S}_{1,3}$. Proceeding in an induction manner, we can assure the existence of a function g belonging to $\mathcal{S}_{1,n}$ for every $n \in \mathbb{N}$. Hence, defining $F : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ by $F := g \circ \pi_1$, i.e.,

$$F(x) = F(x_1, \dots, x_m) = g(x_1) \; , \quad \text{for all } x = (x_1, \dots, x_m) \in \mathbb{R}^m$$

($\pi_1 : \mathbb{R}^m \longrightarrow \mathbb{R}$ denotes the canonical projection over the first coordinate), we conclude that $F \in \mathcal{S}_{m,n}$ (F is composition of continuous surjective functions). □

Attempting maximal lineability of $\mathcal{S}_{m,n}$ (that is, $\mathfrak{c}$-lineability) we make use of the following remark (inspired in a result from [1]), which indicates a method to obtain our main result.

Remark 2.2. *Given a continuous surjection $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$, suppose we have $\mathcal{X} \subset \mathcal{C}(\mathbb{R}^n; \mathbb{R}^n)$ a subset of $\mathfrak{c}$-many linearly independent functions such that every nonzero element of span$(\mathcal{X})$ is a continuous surjection. Then, we have that*

$$\mathcal{Y} := \{F \circ f\}_{F \in \mathcal{X}} \subset \mathcal{C}(\mathbb{R}^m; \mathbb{R}^n)$$

has cardinality $\mathfrak{c}$, is linearly independent and is formed just by continuous surjections. Moreover,

$$\mathit{span}(\mathcal{Y}) \subset \mathcal{S}_{m,n} \cup \{0\},$$

obtaining the $\mathfrak{c}$-lineability of $\mathcal{S}_{m,n}$.

In order to continue we shall need two lemmas and some notation. First, let us consider (for $r > 0$) the homeomorphism $\phi_r : \mathbb{R} \to \mathbb{R}$ given by

$$\phi_r(t) := e^{rt} - e^{-rt}.$$

Lemma 2.3. *The subset $\mathfrak{A} := \{\phi_r\}_{r \in \mathbb{R}^+}$ of $\mathbb{R}^{\mathbb{R}}$ is linearly independent, has cardinality $\mathfrak{c}$, and every nonzero element of span$(\mathfrak{A})$ is continuous and surjective.*

Proof. First let us prove that every nonzero element $\phi = \sum_{i=1}^{k} \alpha_i \cdot \phi_{r_i} \in \text{span}(\mathfrak{A})$ is surjective. We may suppose that $r_1 > r_2 > \cdots > r_k$ and $\alpha_1 \neq 0$. Writing

$$\phi(t) = e^{r_1 t} \cdot \left(\alpha_1 + \sum_{i=2}^{k} \alpha_i \cdot e^{(r_i - r_1)t} \right) - \sum_{i=1}^{k} \alpha_i \cdot e^{-r_i t},$$

we conclude that $\lim_{t \to +\infty} \phi(t) = \text{sign}(\alpha_1) \cdot \infty$ and $\lim_{t \to -\infty} \phi(t) = -\text{sign}(\alpha_1) \cdot \infty$. Thus, the continuity of ϕ assures its surjection. Now let us see that $\mathfrak{A}$ is linearly independent: suppose that $\psi = \sum_{i=1}^{n} \lambda_i \cdot \phi_{s_i} = 0$. If there is some $\lambda_j \neq 0$, we may suppose that $s_1 > \cdots > s_n$ and $\lambda_1 \neq 0$. Repeating the argument above, we obtain

$$\lim_{t \to +\infty} \psi(t) = \text{sign}(\lambda_1) \cdot \infty \;\text{ and }\; \lim_{t \to -\infty} \psi(t) = -\text{sign}(\lambda_1) \cdot \infty,$$

which contradicts $\psi = 0$. This proves that $\mathfrak{A}$ is linearly independent. The other assertions are easy to prove. $\square$

For each $r = (r_1, \ldots, r_n) \in (\mathbb{R}^+)^n$, let φ_r be the homeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$ defined by $\varphi_r = (\phi_{r_1}, \ldots, \phi_{r_n})$, *i.e.*,

$$\varphi_r(x) := (\phi_{r_1}(x_1), \ldots, \phi_{r_n}(x_n)), \quad \text{for all } x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Working on each coordinate, and using the previous lemma, we have the following.

Lemma 2.4. *The set $\mathfrak{B} = \{\varphi_r\}_{r \in (\mathbb{R}^+)^n}$ of $\mathcal{C}(\mathbb{R}^n; \mathbb{R}^n)$ is linearly independent, has cardinality $\mathfrak{c}$, and every nonzero element of span$(\mathfrak{B})$ is continuous and surjective.*

Now it is time to state and prove our main result.

Theorem 2.5. *$\mathcal{S}_{m,n}$ is $\mathfrak{c}$-lineable.*

Proof. Let $f \in \mathcal{S}_{m,n}$. Using the notation of the previous lemma and the ideas of the Remark 2.2, we now prove that the set $\mathfrak{C} = \{F \circ f\}_{F \in \mathfrak{B}}$ is so that span$(\mathfrak{C})$ is the space we are looking for.

The surjectivity of f assures that $G \circ f = 0$ implies $G = 0$, for every function G from $\mathbb{R}^n$ to $\mathbb{R}^n$. Thus, if $G_i \in \mathfrak{B},\ i = 1, \ldots, k$ and

$$0 = \sum_{i=1}^{k} \alpha_i \cdot G_i \circ f = \left(\sum_{i=1}^{k} \alpha_i G_i \right) \circ f,$$

then $\sum_{i=1}^{k} \alpha_i \cdot G_i = 0$; so since $\mathfrak{B}$ is linearly independent, we conclude that $\alpha_i = 0,\ i = 1, \ldots, k$ and thus, $\mathfrak{C}$ is linearly independent. Thus, clearly, it has cardinality $\mathfrak{c}$. Furthermore, any nonzero function

$$\sum_{i=1}^{l} \lambda_i \cdot F_i \circ f = \left(\sum_{i=1}^{l} \lambda_i F_i \right) \circ f$$

of span$(\mathfrak{C})$ is continuous and surjective, since it is the composition of continuous surjective functions (recall that, from Lemma 2.4, $\sum_{i=1}^{l} \lambda_i F_i$ is a continuous surjective function). Therefore, span$(\mathfrak{C})$ only contains, except the zero function, continuous surjective functions. $\square$

Remark 2.6. *As we mentioned in the Introduction, and since $\dim \mathcal{C}(\mathbb{R}^m, \mathbb{R}^n) = \mathfrak{c}$, this result is the best possible in terms of dimension. The next step (in sense of trying a similar result in higher dimensions) could be related to the lineability of $\mathcal{S}_{m,\mathbb{N}}$ (the set of the continuous surjections from $\mathbb{R}$ onto $\mathbb{R}^{\mathbb{N}}$ with the product topology). However this is not possible, since $\mathcal{S}_{m,\mathbb{N}} = \emptyset$ (*[9, p. 275]*).*

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